$\boxed{\text{TODAY: Every statement is valid also in pseudo riemannian case}}$

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Metric connections (Why?)

$\forall X, Y, Z \in \mathfrak{X}(M)$

$$g(X,Y) = (C^1_1 \circ C^1_1)(g \otimes X \otimes Y)$$

$$\Rightarrow \quad \nabla_Z (g(X,Y)) = C^1_1\left(C^1_1\left(\nabla_Z g \otimes X \otimes Y + g \otimes \nabla_Z X \otimes Y + g \otimes X \otimes \nabla_Z Y\right)\right)$$

$$= (\nabla_Z g)(X,Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

$$\boxed{\nabla_Z (g(X,Y)) = (\nabla_Z g)(X,Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y)}$$

Def

∇ is metric iff $\forall Z \in \mathfrak{X}(M)$ $\quad \nabla_Z g \equiv 0$.

Thus for metric connections we have

$$\nabla_Z (g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Corollary

∇ is metric $\Leftrightarrow$ Parallel transport preserves scalar product of vectors.

Proof

$\gamma \rightsquigarrow Z = \frac{d\gamma}{dt}$

Let X, Y be two vectors parallely transported along γ, s.t.

$X(0) = X_0$, $Y(0) = Y_0$. We have $\frac{DX}{dt} = 0$, $\frac{DY}{dt} = 0$

$$\frac{d}{dt}\Big(g(X(t), Y(t))\Big) = \left(\nabla_Z\, g\right)(X(t), Y(t)) + \cancel{g\left(\tfrac{DX}{dt}, Y\right)} + \cancel{g\left(X, \tfrac{DY}{dt}\right)}$$

$\Rightarrow$: if $\nabla_Z g \equiv 0 \;\; \forall Z \Rightarrow \frac{d}{dt}\, g(X(t), Y(t)) = 0$

$\Rightarrow g(X(t), Y(t)) = \text{const}$ along γ.

$\Leftarrow$: if $g(X(t), Y(t)) = \text{const}$ along $\gamma \Rightarrow$

$\nabla_Z g \equiv 0$ along γ

but we want that $g(X(t), Y(t)) = \text{const}$ along all γs

$\Rightarrow \nabla_Z g \equiv 0$ along any curve

$\Rightarrow \nabla_Z g \equiv 0 \;\; \forall Z \in \mathfrak{X}(M)$.

$\square$.

This in particular means that if we calculate in local frames:

$$\nabla_\mu X_\nu = \nabla_\mu \left(g_{\nu\rho} X^\rho\right) = \cancel{\left(\nabla_\mu g\right)_{\nu\rho} X^\rho} + g_{\nu\rho} \nabla_\mu X^\rho$$

if connection is metric we may commute $g_{\nu\rho}$ with ∇_μ!

But <u>only if</u> ∇ is metric!

<u>Riemann tensor</u> $\equiv$ curvature tensor for the (M, g) Levi-Civita connection.

$$\left.\begin{array}{l} (Dg)_{\mu\nu} \equiv 0 \\ \Theta^{\mu} \equiv 0 \end{array}\right\} \Longleftrightarrow \nabla \text{ - Levi-Civita}$$

$$\Updownarrow$$

$$\begin{cases} \nabla_X g \equiv 0 & \forall X \in \mathfrak{X}(M) \\ \nabla_X Y - \nabla_Y X = [X,Y] & \forall X, Y \in \mathfrak{X}(M) \end{cases}$$

$$\boxed{R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.} \qquad \forall X, Y, Z \in \mathfrak{X}(M)$$

$\uparrow$

Riemann tensor.

<u>Symmetries</u>

1) $R(X,Y) = -R(Y,X)$

2) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$ $\qquad$ Ist Bianchi (no torsion!)

3) $g\big(R(X,Y)Z, T\big) = -g\big(R(X,Y)T, Z\big)$

4) $g(R(X,Y)\mathbf{Z}, T) = g(R(Z,T)\mathbf{X}, Y)$

or:

$$R(X_\mu, X_\nu)Y_\rho = R^\sigma{}_{\rho\mu\nu} Y_\sigma \xleftrightarrow{g_{\alpha\beta}} R_{\sigma\rho\mu\nu}$$

1) $R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu}$

2) $R_{\sigma\rho\mu\nu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$

3) $R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu}$

4) $R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}$

Comments

1) holds for <u>any</u> connection

2) holds if torsion vanishes (I^{st} Bianchi $+ T \equiv 0$)

3) holds if connection is metric ($Dg \equiv 0$)

4) holds for Levi-Civita. (i.e $T \equiv 0$ & $Dg \equiv 0$)

(homework: Thursday 23 Oct. prove 3) and 4) using <u>any</u> of the three languages.)

<u>Fact</u> # of independent components of Riemann: $\frac{1}{12} n^2 (n^2 - 1)$

<u>Thm</u>

Riemann $\equiv 0$ $\iff$ there exists a local coord system (x^i) s.t.

$$g = dx^{1^2} + \dots + dx^{p^2} - dx^{p+1^2} - \dots - dx^{n^2}$$

$$(g_{\mu\nu} = \mathrm{diag}(1, \dots 1, -1, \dots -1))$$

<u>Proof</u>

$$\Gamma^\mu{}_\nu = \frac{1}{2} g^{\mu\alpha} (g_{\alpha\nu,\rho} + g_{\rho\alpha,\nu} - g_{\nu\rho,\alpha}) dx^\rho$$

$\Leftarrow$ Hence: $\Gamma^\mu{}_\nu \equiv 0 \Rightarrow \Omega^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu \equiv 0$ ✓

$\Rightarrow$ $R^\mu{}_{\nu\rho\sigma} \equiv 0$, we showed before that this means that one can make $\Gamma^\mu{}_{\nu\rho} \equiv 0$ in a neighbourhood in holonomic frame x^μ.
And because $T^\mu{}_{\nu\rho} \equiv 0$ this can be made in a neighbourhood

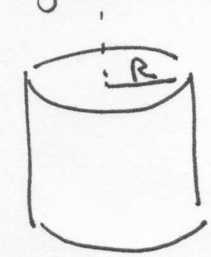
$\Rightarrow$ $g_{\alpha\nu,\rho} = 0$ in a neighbourhood

$\Rightarrow$ $g_{\alpha\nu} = (\mathrm{const})_{\alpha\nu}$

$\Rightarrow$ Linear transf. of coordinates brings $g_{\alpha\nu} = \mathrm{diag}(1, \dots 1, -1, \dots -1)$ $\square$

Example

Cylinder



$$g\Big|_{\text{cylinder}} = dx^2 + dy^2 + dz^2 = dz^2 + R^2 d\varphi^2$$

const

$$\Longrightarrow \quad R^{\mu}{}_{\nu\rho\sigma} = 0 \quad \Longrightarrow \quad \text{cylinder is } \underline{\text{flat}} .$$

Decomposition of Riemann onto irreducible bits

Example

$V = \mathbb{R}^n$,

$A_{\mu\nu} \in (\mathbb{R}^{n*}) \otimes (\mathbb{R}^{n*})$

$G = GL(n, \mathbb{R})$ acts on $V^* \otimes V^*$ via:

$$V^*\otimes V^* \ni A_{\alpha\beta} \xrightarrow{\;a\;} \rho(a)^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} A_{\alpha\beta} = A_{\alpha\beta}\, a^{-1\alpha}{}_{\mu}\, a^{-1\beta}{}_{\nu} \in V^* \otimes V^*$$

Hence we have <u>representation</u> $\rho^{\alpha\beta}{}_{\mu\nu}$ of $GL(n,\mathbb{R})$ in $V^*\otimes V^*$.

But this representation is <u>reducible</u>. Indeed

$V^*\wedge V^*$ and $V^*\odot V^*$ are $GL(n,\mathbb{R})$ invariant subspaces.

$$A_{\alpha\beta} = A_{(\alpha\beta)} + A_{[\alpha\beta]}$$

$$A_{(\alpha\beta)} = \tfrac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha}) \in V^*\odot V^*$$

$$A_{[\alpha\beta]} = \tfrac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha}) \in V^*\wedge V^*$$

$$V^*\otimes V^* = V^*\odot V^* \oplus V^*\wedge V^*$$

± 1 eigenspaces of S s.t.

$S A_{\alpha\beta} = A_{\beta\alpha}$.

$\rho(a)^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} A_{(\alpha\beta)} = A'_{\mu\nu}$ is symmetric

$\rho(a)^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} A_{[\alpha\beta]} = A'_{\mu\nu}$ is antisymmetric !

This is the end! Spaces $V^* \odot V^*$ and $V^* \wedge V^*$ are irreducible w.r.t. $GL(n,\mathbb{R})$.

Suppose now that we in addition have (pseudo) riemannian <u>metric</u> $g_{\mu\nu}$ in $V = \mathbb{R}^n$.

We restrict the group $GL(n,\mathbb{R})$ to its subgroup $O(g)$ preserving metric:

$$O(g) = \{ a \in GL(n,\mathbb{R}) \text{ s.t. } g_{\mu\nu} a^{-1\mu}{}_{\alpha} a^{-1\mu}{}_{\beta} = g_{\alpha\beta} \}.$$

What about decomposition of $V^* \otimes V^*$ on irreducibles w.r.t. the <u>restricted</u> group $O(g)$?

We have

$$V^* \otimes V^* \underset{\substack{\uparrow \\ O(g)}}{=} V^* \wedge V^* \oplus \underbrace{V^* \odot V^*}_{\substack{\uparrow \\ \text{but now this is reducible!}}}$$

It has an invariant subspace

$$Tr = \{ A_{\mu\nu} = \lambda g_{\mu\nu} ,\ \lambda \in \mathbb{R} \} \subset V^* \odot V^*$$

$$V^* \odot V^* = (V^* \odot V^*)_0 \oplus Tr(V^* \odot V^*)$$

We have $A_{\mu\nu}$, $g_{\mu\nu}$ and its inverse $g^{\alpha\beta}$ s.t.

$$g_{\alpha\beta}\, g^{\beta\nu} = \delta_\alpha{}^\nu .$$

$$A_{(\mu\nu)} \rightsquigarrow A = g^{\mu\nu} A_{\mu\nu}$$

$$A_{(\mu\nu)} \rightsquigarrow \check{A}_{(\mu\nu)} = A_{(\mu\nu)} - \frac{1}{n} A\, g_{\mu\nu}$$

is traceless.

$$\boxed{A_{\mu\nu} = \underset{\substack{\uparrow \\ V^* \wedge V^*}}{A_{[\mu\nu]}} + \underset{\substack{\uparrow \\ (V^* \odot V^*)_0}}{\check{A}_{(\mu\nu)}} + \underset{\substack{\uparrow \\ \operatorname{Tr}(V^*\odot V^*)}}{\frac{1}{n} A\, g_{\mu\nu}}}$$

With the exception of $\dim V = 4$ (+ orientability) this is decomposition onto irreducibles w.r.t. $O(g)$.

The same for Riemann:

$$R^\mu{}_{\nu\rho\sigma} \longrightarrow \underset{\substack{\uparrow \\ \text{is symmetric}}}{R_{\nu\sigma} = R^\mu{}_{\nu\mu\sigma}}$$

This is called RICCI tensor

$$R_{\nu\sigma} \xrightarrow{\operatorname{Tr}} R = g^{\nu\sigma} R_{\nu\sigma}$$

This is called RICCI scalar / SCALAR CURVATURE

$$\check{R}_{\nu\sigma} = R_{\nu\sigma} - \frac{1}{n} R\, g_{\nu\sigma}$$

$$R^{\mu\nu}{}_{\rho\sigma} = \underset{\text{totally traceless.}}{\boxed{C^{\mu\nu}{}_{\rho\sigma}}} + a\, \delta^{[\mu}_{[\rho}\, \check{R}^{\nu]}_{\sigma]} + b\, \delta^{[\mu}_{\rho}\, \delta^{\nu]}_{\sigma}\, R$$

Calculate a and b.

1) contraction over μ, ρ:

$$R^{\nu}{}_{\sigma} = \frac{a}{4}\left(n \check{R}^{\nu}{}_{\sigma} - \check{R}^{\nu}{}_{\sigma} - \check{R}^{\nu}{}_{\sigma} + \cancel{\delta^{\nu}{}_{\sigma} \check{R}}\right)$$
$$+ \frac{b}{2}(n-1)\,\delta^{\nu}{}_{\sigma} R$$

But always:

$$R^{\nu}{}_{\sigma} = \check{R}^{\nu}{}_{\sigma} + \frac{1}{n}\delta^{\nu}{}_{\sigma} R$$

hence:

$$\frac{a}{4}(n-2) = 1 \quad \Rightarrow \quad a = \frac{4}{n-2}$$
$$\frac{b}{2}(n-1) = +\frac{1}{n} \quad \Rightarrow \quad b = \frac{-2}{n(1-n)}$$

$$\boxed{R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + \frac{4}{n-2}\,\delta^{[\mu}_{[\rho}\,\check{R}^{\nu]}{}_{\sigma]} + \frac{2}{n(n-1)}\,\delta^{[\mu}_{\rho}\,\delta^{\nu]}_{\sigma}\,R}$$

Weyl tensor.

$$\frac{1}{12}n^2(n^2-1) \xrightarrow{n=1} 0, \quad \xrightarrow{n=2} 1, \quad \xrightarrow{n=3} 6, \quad \xrightarrow{n=4} 20$$

Low dimensions

$n=1 \Rightarrow R_{\mu\nu\rho\sigma} \equiv 0$

$n=2 \Rightarrow R_{\mu\nu\rho\sigma}$ has only <u>one</u> component

$\Rightarrow C_{\mu\nu\rho\sigma} \equiv 0,\ \check{R}_{\mu\nu} \equiv 0 \Rightarrow$ only $R \neq 0$.

$n=3 \Rightarrow$ <u>six</u> components $\Rightarrow C_{\mu\nu\rho\sigma} \equiv 0$. All curvature in $R_{\mu\nu}$

$n \geqslant 4$ in general $C_{\mu\nu\rho\sigma} \not\equiv 0$.

Def

Two metrics g and $\hat{g}$ are conformally equivalent iff there exists $r : M \to \mathbb{R}$ s.t.

$$\hat{g} = e^{2r} g.$$

Thm

1) $\hat{C}^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma}$ for conformally equivalent metrics.

2) $n \geq 4$ a metric g is conformally equivalent to a ~~u~~flat metric iff $C^{\mu}{}_{\nu\rho\sigma} \equiv 0$.

Analogous fact for Riemann:

2) $R^{\mu}{}_{\nu\rho\sigma} \equiv 0 \iff$ metric g is isometric to flat metric

Isometry

$$(M, g) \xrightarrow{\varphi} (M', g')$$

$$\varphi^* g' = g$$

1) $\varphi^* R' = R$